# First-Order Definable Counting-Only Queries

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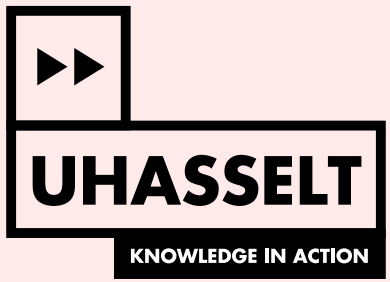


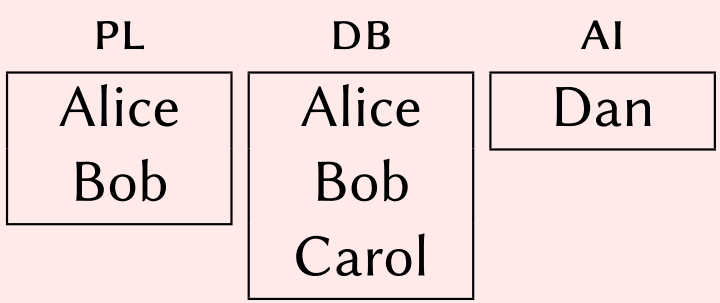
# Bag-of-sets datasets and queries

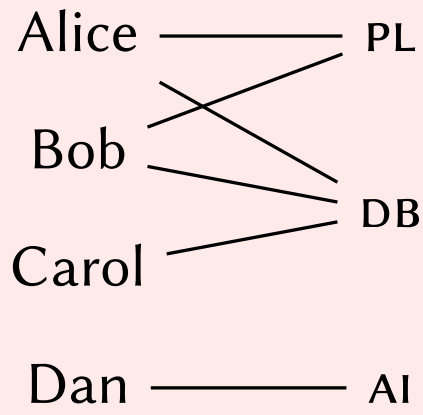
- ‘Return students who take at least 2 courses.’

$$\{\langle x \rangle \mid \mathsf{count}(x) \geq 2\}$$

- ‘Return pairs of students who take the same number of courses.’

$$\{\langle x, y \rangle \mid (x \neq y) \wedge \mathsf{count}(x) = \mathsf{count}(y)\}$$



# Formalization: the bag-of-sets data model

## Definition

A *structure* $\mathbf{S}$ over domain $\mathcal{D}$ of objects is a pair $\mathbf{S} = (\mathbf{N}, \gamma)$, with:

- $\mathbf{N}$ a finite set of *set names*;
- $\gamma \subset \mathcal{D} \times \mathbf{N}$ a finite *set-membership* relation.

Let $\mathbf{S} = (\mathbf{N}, \gamma)$, $\mathsf{N} \in \mathbf{N}$ a set name, and $\mathsf{A} \subset \mathcal{D}$ a finite set of objects:

- $\mathsf{objects}(\mathsf{N}; \mathbf{S}) = \{\mathsf{o} \mid (\mathsf{o}, \mathsf{N}) \in \gamma\}$.
- The *cover* is defined by

$$\mathsf{cover}(\mathsf{A}; \mathbf{S}) = \{\mathsf{N} \mid (\mathsf{N} \in \mathbf{N}) \wedge (\mathsf{A} \subseteq \mathsf{objects}(\mathsf{N}; \mathbf{S}))\}.$$

- The *support* is defined by

$$[\![\mathsf{count}(\mathsf{A})]\!]_{\mathbf{S}} = |\mathsf{cover}(\mathsf{A}; \mathbf{S})|.$$

# Counting-only queries

- ‘Return pairs of distinct students which take a common course.’

$$\{\langle x, y\rangle \mid (x \neq y) \wedge \mathsf{count}(x, y) \geq 1\}$$

- ‘Return pairs of distinct students which take the same courses.’

$$\{\langle x, y\rangle \mid (x \neq y) \wedge \mathsf{count}(x, y) = \mathsf{count}(x) = \mathsf{count}(y)\}$$

| PL | DB | AI | VR |
|---|---|---|---|
| Alice<br>Bob | Alice<br>Carol | Bob<br>Carol | |

| PL | DB | AI | VR |
|---|---|---|---|
| Alice<br>Bob<br>Carol | Alice | Bob | Carol |

$$\begin{aligned}
\mathsf{count}() &= 4;\\
\mathsf{count}(\mathrm{A}) &= 2;\\
\mathsf{count}(\mathrm{B}) &= 2;\\
\mathsf{count}(\mathrm{C}) &= 2;\\
\mathsf{count}(\mathrm{A}, \mathrm{B}) &= 1;\\
\mathsf{count}(\mathrm{A}, \mathrm{C}) &= 1;\\
\mathsf{count}(\mathrm{B}, \mathrm{C}) &= 1.
\end{aligned}$$

# Formalization: $k$-counting-only queries

## Definition

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two structures.

- $\mathbf{S}_1$ and $\mathbf{S}_2$ are *exactly-$k$-counting-equivalent* if, for every set of objects A with $|\mathrm{A}| = k$,

$$[\![\mathrm{count}(\mathrm{A})]\!]_{\mathbf{S}_1} = [\![\mathrm{count}(\mathrm{A})]\!]_{\mathbf{S}_2}.$$

- $\mathbf{S}_1$ and $\mathbf{S}_2$ are *$k$-counting-equivalent* if they are exactly-$j$-counting-equivalent for every $j$, $0 \leq j \leq k$.

## Definition

Let $\mathtt{Q}$ be a query.

- $\mathtt{Q}$ is *$k$-counting-only* if $[\![\mathtt{Q}]\!]_{\mathbf{S}_1} = [\![\mathtt{Q}]\!]_{\mathbf{S}_2}$ for every pair of $k$-counting-equivalent structures $\mathbf{S}_1$ and $\mathbf{S}_2$.
- $\mathtt{Q}$ is *counting-only* if it is $k$-counting-only for some $k$.

# Proving that a query is not 2-counting-only

'Does there exist a course taken by 3 students?'

$$\{\langle\rangle \mid \exists x \exists y \exists z\, ((x \neq y \wedge x \neq z \wedge y \neq z) \wedge \mathsf{count}(x, y, z) \geq 1)\}$$

This query is clearly 3-counting-only

## Is this query 2-counting-only?

It can distinguish between 2-counting-equivalent structures!

| PL | DB | AI | VR |
|---|---|---|---|
| Alice<br>Bob | Alice<br>Carol | Bob<br>Carol | |

| PL | DB | AI | VR |
|---|---|---|---|
| Alice<br>Bob<br>Carol | Alice | Bob | Carol |

# Proving that a query is 2-counting-only

'Does there exists a student who takes courses that are taken by a pair of other students?'

$$\{\langle\rangle \mid \exists x \exists y_1 \exists y_2\ (x \neq y_1) \wedge (x \neq y_2) \wedge (y_1 \neq y_2) \wedge \\ \mathsf{count}(y_1) = \mathsf{count}(x, y_1) \wedge \mathsf{count}(y_2) = \mathsf{count}(x, y_2) \wedge \\ \mathsf{count}(x) = \mathsf{count}(x, y_1) + \mathsf{count}(x, y_2) - \mathsf{count}(x, y_1, y_2)\}.$$

This query is clearly 3-counting-only.

## Is this query 2-counting-only?

It is equivalent to the 2-counting-only query

$$\{\langle\rangle \mid \exists x \exists y_1 \exists y_2\ (x \neq y_1) \wedge (x \neq y_2) \wedge (y_1 \neq y_2) \wedge \\ \mathsf{count}(y_1) = \mathsf{count}(x, y_1) \wedge \mathsf{count}(y_2) = \mathsf{count}(x, y_2) \wedge \\ \mathsf{count}(x) = \mathsf{count}(x, y_1) + \mathsf{count}(x, y_2) - \mathsf{count}(y_1, y_2)\}.$$

# First-order logic and counting-only queries

# SyCALC: first-order logic on bag-of-sets

## Definition

*SyCALC formulae* are defined by the grammar

$$e := \Gamma(x, X) \mid x = y \mid X = Y \mid e \vee e \mid \neg e \mid \exists x\ e \mid \exists X\ e.$$

A *SyCALC query* is a formula without free set name variables.

## Example

- 'Return students who take at least 2 courses.'

$$\{\langle x \rangle \mid \mathsf{count}(x) \geq 2\} \quad \longrightarrow \quad \{\langle x \rangle \mid \exists Y \exists Z\ ((Y \neq Z) \wedge \Gamma(x, Y) \wedge \Gamma(x, Z))\}$$

- 'Return pairs of distinct students which take a common course.'

$$\{\langle x, y \rangle \mid (x \neq y) \wedge \mathsf{count}(x, y) \geq 1\} \quad \longrightarrow \quad \{\langle x, y \rangle \mid (x \neq y) \wedge \exists C\ (\Gamma(x, C) \wedge \Gamma(y, C))\}$$

# $k$-SyCALC: $k$-counting-only SyCALC

**Definition**

$k$-SyCALC, $k \geq 0$, denotes the *$k$-counting-only SyCALC queries*.

**Proposition**

*Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be $k$-SyCALC queries:*

- $\mathbf{Q}_1 \vee \mathbf{Q}_2$ *is in $k$-SyCALC.*
- $\neg\mathbf{Q}_1$ *is in $k$-SyCALC.*
- $\exists x\, \mathbf{Q}_1$ *is in $k$-SyCALC.*

Hence: $k$-SyCALC is closed under disjunction, negation, and object quantification.

# Not all counting-only queries are in SyCALC

- ‘Return pairs of students who take the same number of courses.’

$$\{\langle x, y\rangle \mid (x \neq y) \wedge \mathsf{count}(x) = \mathsf{count}(y)\}.$$

  This query is 1-counting-only, but not first-order definable.

- ‘Are there an even number of courses?’

$$\{\langle\rangle \mid \exists k \ \mathsf{count}() = 2k\}.$$

  This query is 0-counting-only, but not first-order definable.

# Not all SyCALC queries are counting-only

- ‘Does there exist a course followed by all students?’

$$\{\langle\rangle \mid \exists C \forall x \exists Y \, (\Gamma(x, Y) \implies \Gamma(x, C))\}$$

- ‘Does there exist a course followed by no students?’

$$\{\langle\rangle \mid \exists C \forall x \, (\neg\Gamma(x, C))\}$$

## How to prove this?

Find $k$-counting equivalent structures distinguished by these queries (for every $k$).

# Constructing $k$-counting equivalent structures

Example: $k = 3$.

| $S_1$ |
| --- |
| A |
| B |
| C |
| D |

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Example: $k = 3$.

| $S_1$ |
|---|
| A |
| B |
| C |
| D |

| $T_1$ | $T_2$ | $T_3$ | $T_4$ |
|---|---|---|---|
| A | A | A | B |
| B | B | C | C |
| C | D | D | D |

# Constructing $k$-counting equivalent structures

Example: $k = 3$.

| $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ |
|---|---|---|---|---|---|---|
| A | A | A | A | B | B | C |
| B | B | C | D | C | D | D |
| C | | | | | | |
| D | | | | | | |

| $T_1$ | $T_2$ | $T_3$ | $T_4$ |
|---|---|---|---|
| A | A | A | B |
| B | B | C | C |
| C | D | D | D |

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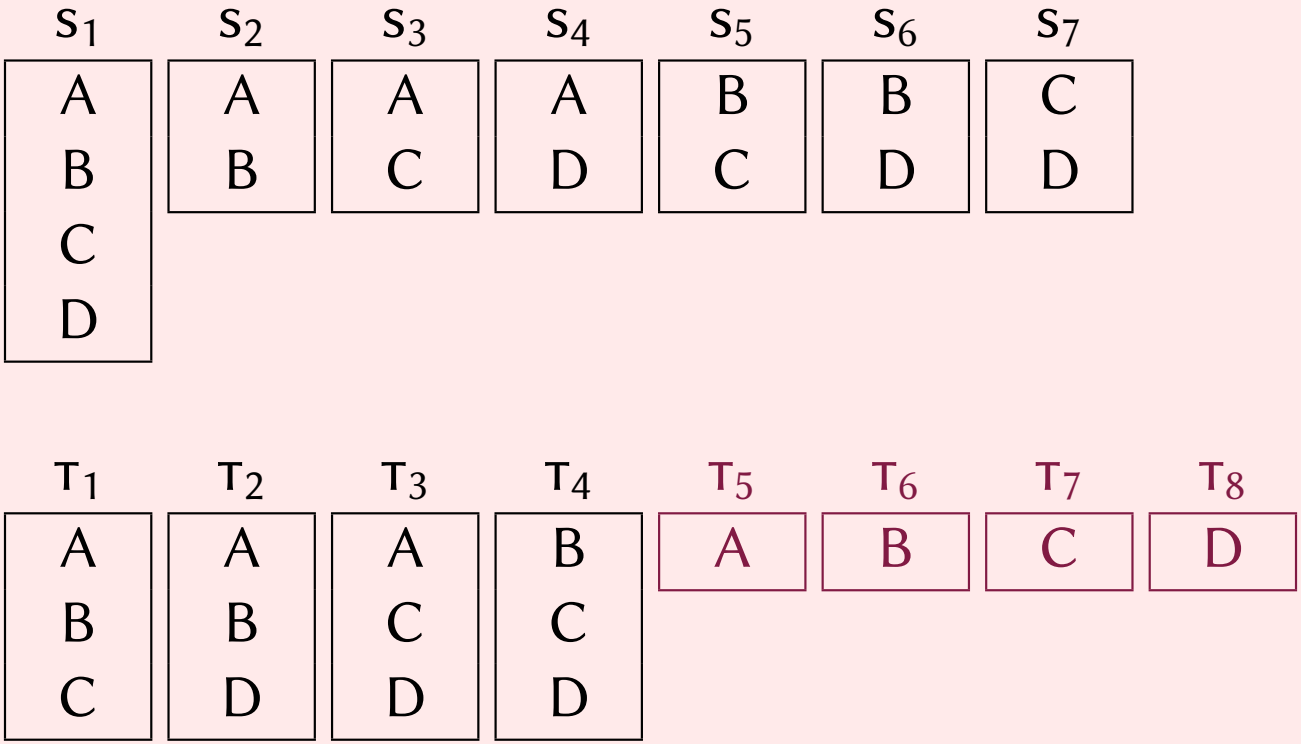

# Constructing $k$-counting equivalent structures

Example: $k = 3$.

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ | $s_8$ |
|---|---|---|---|---|---|---|---|
| A | A | A | A | B | B | C | |
| B | B | C | D | C | D | D | |
| C | | | | | | | |
| D | | | | | | | |

| $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ | $t_8$ |
|---|---|---|---|---|---|---|---|
| A | A | A | B | A | B | C | D |
| B | B | C | C | | | | |
| C | D | D | D | | | | |

# Constructing $k$-counting equivalent structures

Example: $k = 3$.

| $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |
|---|---|---|---|---|---|---|---|
| A | A | A | A | B | B | C | |
| B | B | C | D | C | D | D | |
| C | | | | | | | |
| D | | | | | | | |

| $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ |
|---|---|---|---|---|---|---|---|
| A | A | A | B | A | B | C | D |
| B | B | C | C | | | | |
| C | D | D | D | | | | |

Structures are 3-counting equivalent, but not 4-counting equivalent.

# A counting-only fragment of SyCALC

# A powerful observation

Many queries can be expressed using simple *generalized count* terms.

## Example

'Does there exists a student who takes courses that are taken by a pair of other students?'

$$\{\langle\rangle \mid \exists x \exists y_1 \exists y_2\ (x \neq y_1) \wedge (x \neq y_2) \wedge (y_1 \neq y_2) \wedge \\ \mathsf{count}(y_1) = \mathsf{count}(x, y_1) \wedge \mathsf{count}(y_2) = \mathsf{count}(x, y_2) \wedge \\ \mathsf{count}(x) = \mathsf{count}(x, y_1) + \mathsf{count}(x, y_2) - \mathsf{count}(x, y_1, y_2)\}.$$

This query is equivalent to

$$\{\langle\rangle \mid \exists x \exists y_1 \exists y_2\ (x \neq y_1) \wedge (x \neq y_2) \wedge (y_1 \neq y_2) \wedge \\ \mathsf{gcount}(x; y_1, y_2) = 0 \wedge \mathsf{gcount}(y_1; x) = 0 \wedge \mathsf{gcount}(y_2; x) = 0\}.$$

# QuineCALC and SimpleCALC

Terms $\mathrm{count}(\mathcal{S}) \geq c$ or $\mathrm{gcount}(\mathcal{X}; \mathcal{Y}) \geq c$ are simple to express:

$$\mathrm{gcount}(\mathcal{X}; \mathcal{Y}) \geq c = \exists Z_1 \ldots \exists Z_c$$
$$\Big( \bigwedge_{1 \leq i < j \leq c} (Z_i \neq Z_j) \wedge \bigwedge_{x \in \mathcal{X}} \big(\Gamma(x, Z_1) \wedge \cdots \wedge \Gamma(x, Z_c)\big) \wedge$$
$$\bigwedge_{y \in \mathcal{Y}} \big(\neg\Gamma(y, Z_1) \wedge \cdots \wedge \neg\Gamma(y, Z_c)\big)\Big).$$

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$$\bigwedge_{y \in \mathcal{Y}} \big(\neg\Gamma(y, Z_1) \wedge \cdots \wedge \neg\Gamma(y, Z_c)\big) \Big).$$

## Definition

- QuineCALC-$k$ consist of all SyCALC queries that do not use object quantification and with at most $k$ free object variables.
- SimpleCALC-$k$ consists of all queries that are built from QuineCALC-$k$ queries using disjunction, negation, and object quantification.

# Every SimpleCALC-$k$ query is in $k$-SyCALC

## Proposition

*Every SimpleCALC-k query is k-counting-only.*

- SimpleCALC-$k$ are built from QuineCALC-$k$ queries using disjunction, negation, and object quantification.
- Closure results of $k$-SyCALC apply to SimpleCALC-$k$.

Hence: prove that QuineCALC-$k$ queries are $k$-counting-only!

# Every QuineCALC-$k$ query is in $k$-SyCALC

## Definition

Let $\mathbf{S} = (\mathbf{N}, \gamma)$ be a structure. If A is a set of objects, then $\mathbf{S}|_{\mathrm{A}}$ denotes the structure $(\mathbf{N}, \gamma \cap (\mathrm{A} \times \mathbf{N}))$.

## Lemma

*Let* $\mathbf{S} = (\mathbf{N}, \gamma)$ *be a structure and* $\mathbf{Q}(x_1, \ldots, x_k)$ *be a QuineCALC-k query. We have*

$$\langle \mathrm{o}_1, \ldots, \mathrm{o}_k \rangle \in [\![\mathbf{Q}]\!]_{\mathbf{S}} \iff \langle \mathrm{o}_1, \ldots, \mathrm{o}_k \rangle \in [\![\mathbf{Q}]\!]_{\mathbf{S}|_{\{\mathrm{o}_1, \ldots, \mathrm{o}_k\}}}.$$

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$$\langle o_1, \ldots, o_k \rangle \in [\![\mathbf{Q}]\!]_{\mathbf{S}} \iff \langle o_1, \ldots, o_k \rangle \in [\![\mathbf{Q}]\!]_{\mathbf{S}|_{\{o_1, \ldots, o_k\}}}.$$

## Lemma

*Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be two k-counting-equivalent structures. If* A *is a set of objects with* $|\mathsf{A}| \leq k$*, then* $\mathbf{S}_1|_{\mathsf{A}}$ *is isomorphic to* $\mathbf{S}_2|_{\mathsf{A}}$*.*

# A counting-only hierarchy

## Theorem

*Let $k \geq 0$. There are*

1. *QuineCALC-(k+1) queries and*
2. *Boolean SimpleCALC-(k+1) queries*

*that are not k-counting-only.*

## Proof.

1. 'Return $k+1$ objects that occur together.'

$$\{\langle x_1, \ldots, x_{k+1}\rangle \mid \mathsf{count}(x_1, \ldots, x_{k+1}) \geq 1)\} = \{\langle x_1, \ldots, x_{k+1}\rangle \mid \exists X \left( \bigwedge_{1 \leq i \leq k+1} \Gamma(x_i, X)\right)\}$$

2. 'Does there exist a set with $k+1$ objects?'

$$\exists x_1 \ldots x_{k+1} \left( \left( \bigwedge_{1 \leq i < j \leq k+1} (x_i \neq x_j)\right) \wedge \mathsf{count}(x_1, \ldots, x_{k+1}) \geq 1\right).$$

# Limits to SimpleCALC-$k$

## Proposition

*Let* $\mathbf{Q}(x_1, \ldots, x_m)$ *be a SimpleCALC-k query. There exists an n such that*

$$\langle o_1, \ldots, o_k \rangle \in [\![\mathbf{Q}]\!]_{\mathbf{S}} \iff \langle o_1, \ldots, o_k \rangle \in [\![\mathbf{Q}]\!]_{\mathbf{S}|_A}$$

*for some* A *with* $|A| \leq n$ *and* $\{o_1, \ldots, o_k\} \subseteq A$.

## Example

'Return students that take courses with another student.'

$$\mathbf{Q} = \{\langle x \rangle \mid \exists y \exists C \, (\Gamma(x, C) \wedge \Gamma(y, C))\}$$

# Not all 2-SyCALC queries are in SimpleCALC

'Is each course followed by a unique student?'

$$\texttt{set-ids} = \{\langle\rangle \mid \forall C \exists x\, (\Gamma(x, C) \wedge \neg \exists Y\, ((X \neq Y) \wedge \Gamma(x, Y)))\}$$

## Proposition

*Query* `set-ids` *is* 2*-counting-only, but not* 1*-counting-only.*

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## Proposition

*Query* `set-ids` *is* 2*-counting-only, but not* 1*-counting-only.*

## Proof.

Assume we have $n$ set names. The query `set-ids` is equivalent to

$$\{\langle\rangle \mid \exists x_1 \ldots \exists x_n\, (\bigwedge_{1 \leq i \leq n} \mathsf{count}(x_j) = 1) \wedge (\bigwedge_{1 \leq i < j \leq n} \mathsf{gcount}(x_i; x_j) = 0)\}.$$

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*Query* `set-ids` *is 2-counting-only, but not 1-counting-only.*

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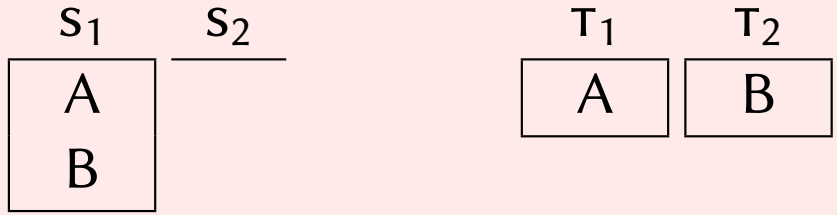
Not 1-counting only:



# Decision problems and counting-only queries

# Classical decision problems

- Satisfiability.
- Validity.
- Query containment.
- Query equivalence.

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# Satisfiability and first-order logic

Satisfiability of first-order logic is *not decidable*.

Satisfiability is decidable for FO when *restricted* to

- unary predicates (monadic first-order logic);
- two variables ($\mathrm{FO}^2$);
- formulae of the form
  - $\exists \cdots \exists \forall \exists \cdots \exists$ (the Ackermann class);
  - $\exists \cdots \exists \forall \forall \exists \cdots \exists$ (the Gödel class);
  - $\exists \cdots \exists \forall \cdots \forall$ (the Schönfinkel-Bernays class).

What if we restrict FO to counting-only queries?

# Satisfiability of 2-SyCALC is undecidable 1/3

FO is undecidable when querying undirected unlabeled graphs without self-loops.

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FO is undecidable when querying undirected unlabeled graphs without self-loops.

## Encode graphs as structures

Encode graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ as the structure $\text{enc}(\mathbf{G}) = (\mathbf{V}, \gamma)$ with

$$\gamma = \{(\{m, n\}, m), (\{m, n\}, n) \mid (m, n) \in \mathbf{E}\} \cup \{(\{n\}, n) \mid n \in \mathbf{V}\}.$$

$\text{enc}(\mathbf{G})$ always satisfies the SyCALC query:

$$\texttt{enc-graph} = \texttt{set-ids} \wedge \forall e \exists M \exists N \left(((M \neq N) \wedge \Gamma(e, M) \wedge \Gamma(e, N)) \Longrightarrow \forall X \left((M \neq X) \wedge (N \neq X) \Longrightarrow \neg\Gamma(e, X)\right)\right).$$

# Satisfiability of 2-SyCALC is undecidable 2/3

FO is undecidable when querying undirected unlabeled graphs without self-loops.

## Encode FO queries as SyCALC queries

Encode Boolean FO query $\varphi$ by $\mathsf{enc}(\varphi) = \mathsf{enc\text{-}graph} \wedge \tau(\varphi)$ with

$$\begin{aligned}
\tau(M = N) &\equiv M = N;\\
\tau(\mathbf{E}(M, N)) &\equiv (M \neq N) \wedge \exists e\, (\Gamma(e, M) \wedge \Gamma(e, N));\\
\tau(e_1 \vee e_2) &\equiv \tau(e_1) \vee \tau(e_2);\\
\tau(\neg e) &\equiv \neg\tau(e);\\
\tau(\exists N\ e) &\equiv \exists N\ \tau(e).
\end{aligned}$$

## Lemma

*Let* $\mathbf{G}$ *be a graph and let* $\varphi$ *be a Boolean FO query. Then,* $[\![\varphi]\!]_{\mathbf{G}} = [\![\mathsf{enc}(\varphi)]\!]_{\mathsf{enc}(\mathbf{G})}$.

# Satisfiability of 2-SyCALC is undecidable 3/3

FO is undecidable when querying undirected unlabeled graphs without self-loops.

## Proposition

*If $\varphi$ is a Boolean FO query, then* $\mathsf{enc}(\varphi)$ *is a 2-SyCALC query.*

## Proof.

If $\mathbf{S}_1$ and $\mathbf{S}_2$ are structures that are 2-counting-equivalent, and $[\![\texttt{set-ids}]\!]_{\mathbf{S}_1} = [\![\texttt{set-ids}]\!]_{\mathbf{S}_2} = \texttt{true}$, then $\mathbf{S}_1$ and $\mathbf{S}_2$ are isomorphic.

## Lemma

*Let $\varphi$ be a Boolean FO query. If there exists a structure $\mathbf{S}$ satisfying* $\mathsf{enc}(\varphi)$*, then we can construct from $\mathbf{S}$ a graph satisfying $\varphi$.*

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## Lemma

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## Theorem

*The satisfiability problem is undecidable for 2-SyCALC queries.*

# What about weaker counting-only languages?

- 1-SyCALC.
- SimpleCALC.
- (QuineCALC).

# What about weaker counting-only languages?

- 1-SyCALC.
- SimpleCALC.
- (QuineCALC).

These classes have *the finite model property*.

# Satisfiability of 1-SyCALC is decidable 1/2

## Definition

Let $k, d \geq 0$. Structures $\mathbf{S}_1 = (\mathbf{N}_1, \gamma_1)$ and $\mathbf{S}_2 = (\mathbf{N}_2, \gamma_2)$ are *$d$-partial $k$-counting-equivalent* if, for every pair of sets of objects I and E with $|\mathsf{I} \cup \mathsf{E}| \leq k$, either

1. $[\![\mathsf{gcount}(\mathsf{I}; \mathsf{E})]\!]_{\mathbf{S}_1} = [\![\mathsf{gcount}(\mathsf{I}; \mathsf{E})]\!]_{\mathbf{S}_2} \leq d$; or
2. $d < [\![\mathsf{gcount}(\mathsf{I}; \mathsf{E})]\!]_{\mathbf{S}_i} < |\mathbf{N}_i| - d$, $i \in \{1, 2\}$,
3. $|\mathbf{N}_1| - [\![\mathsf{gcount}(\mathsf{I}; \mathsf{E})]\!]_{\mathbf{S}_1} = |\mathbf{N}_2| - [\![\mathsf{gcount}(\mathsf{I}; \mathsf{E})]\!]_{\mathbf{S}_2} \leq d$.

## Lemma

*Let $\mathbf{Q}$ be a SyCALC query with set name quantifier depth $d$ and let $\mathbf{S}_1$ and $\mathbf{S}_2$ be $d$-partial $k$-counting-equivalent structures with $k = |\mathsf{adom}(\mathbf{S}_1)| = |\mathsf{adom}(\mathbf{S}_2)|$. Then $[\![\mathbf{Q}]\!]_{\mathbf{S}_1} = [\![\mathbf{Q}]\!]_{\mathbf{S}_2}$.*

# Satisfiability of 1-SyCALC is decidable 2/2

## Proposition

*Let $d \geq 0$, and let $\mathbf{S} = (\mathbf{N}, \gamma)$ be a structure. There exists a structure $\mathbf{S}' = (\mathbf{N}', \gamma')$ with $|\mathbf{N}'| \leq 2d + 1$ such that $\mathbf{S}$ and $\mathbf{S}'$ are $d$-partial 1-counting-equivalent structures.*

## Proposition

*Let $\mathbf{Q}$ be a 1-SyCALC query with set name quantifier depth $d$ and object quantifier depth $r$, and let $\mathbf{S} = (\mathbf{N}, \gamma)$ be a structure. Then, $[\![e]\!]_{\mathbf{S}} \neq \emptyset$ if and only if there exists a structure $\mathbf{S}' = (\mathbf{N}', \gamma')$ with $|\mathbf{N}'| \leq 2d + 1$, $|\mathsf{adom}(\mathbf{S}')| \leq r(2d + 1)$, and $[\![e]\!]_{\mathbf{S}'} \neq \emptyset$.*

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## Theorem

*The satisfiability problem is decidable for 1-SyCALC queries.*

# Satisfiability of SimpleCALC is decidable

## Proposition (Reminder)

*Let* $\mathbf{Q}(x_1, \ldots, x_m)$ *be a SimpleCALC-k query. There exists an n such that*

$$\langle o_1, \ldots, o_k \rangle \in [\![\mathbf{Q}]\!]\mathbf{s} \iff \langle o_1, \ldots, o_k \rangle \in [\![\mathbf{Q}]\!]\mathbf{s}_{|A}$$

*for some* A *with* $|A| \leq n$ *and* $\{o_1, \ldots, o_k\} \subseteq A$.

## Proposition

*Let* $\mathbf{S} = (\mathbf{N}, \gamma)$ *be a structure with* $|\mathsf{adom}(\mathbf{S})| = z$*, and let* $d \geq 0$*. There exists a structure* $\mathbf{S}' = (\mathbf{N}', \gamma')$ *with* $|\mathbf{N}'| \leq (d+1) \cdot 2^z$ *such that* $\mathbf{S}$ *and* $\mathbf{S}'$ *are d-partial z-counting-equivalent structures.*

# Satisfiability of SimpleCALC is decidable

## Proposition (Reminder)

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## Proposition

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## Theorem

*Satisfiability is decidable for SimpleCALC queries, and is NEXPTIME-hard for SimpleCALC-k queries,* $k \geq 2$.

# Conclusion and discussion

# An overview of our results

| First-order definable queries (SyCALC) | | | |
|---|---|---|---|
| QuineCALC | SimpleCALC | c.-o. SyCALC | c.-o. queries |
| ⋮ | ⋮ | ⋮ | ⋮ |
| QuineCALC-3 | SimpleCALC-3 | 3-SyCALC | 3-counting-only |
| QuineCALC-2 | SimpleCALC-2 | 2-SyCALC | 2-counting-only |
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# Future work: generalize bag-of-sets

- ‘Return students that take courses offered by two departments.’

$$\{\langle x \rangle \mid \exists C \exists D_1 \exists D_2 \, (\mathtt{SC}(x, C) \wedge \mathtt{CD}(C, D_1) \wedge \mathtt{CD}(C, D_2))\}$$

- ‘Return student-department pairs in which the student only takes courses offered by that department.’

$$\{\langle x, y \rangle \mid |\{z \mid \mathtt{SC}(x, z) \wedge \mathtt{DC}(y, z)\}| = |\{z \mid \mathtt{SC}(x, z)\}|\}$$

# Future work

- Are SimpleCALC-$k$ and '$\mathsf{gcount}(\mathcal{X}; \mathcal{Y}) \geq c$'-queries equivalent? E.g. 'Return pairs of distinct students which take the same courses.'

$$\{\langle x, y\rangle \mid (x \neq y) \wedge \mathsf{gcount}(x; y) = \mathsf{gcount}(y; x) = 0\}$$

- Decision problems: is a SimpleCALC-$k$ query $l$-counting-only? E.g. 'Are there at least two students taking courses?'

$$\{\langle\rangle \mid \exists x \exists y \exists X \exists Y\, ((x \neq y) \wedge \Gamma(x, X) \wedge \Gamma(y, Y))\}$$

- Possibilities for query optimization?